DISCRETE RADIAN GEOMETRY AS A BOUNDARY–REFINEMENT OF REGGE CALCULUS

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ABSTRACT

Regge calculus provides a coordinate–free discretisation of curvature in which geometric information is concentrated in deficit angles at simplicial hinges [Regge(1961)]. Classical formulations of the theory treat angular contributions per cycle, thereby introducing implicit factors of 2π into curvature, action, and holonomy. This paper develops a boundary–refined variant of Regge calculus in which all angular quantities are normalised per radian, eliminating these hidden conventions and providing a more transparent geometric foundation.

The refinement is driven by the existence of a minimal non–contractible geodesic on a compact boundary, guaranteed by standard results in Riemannian geometry [Besse(1978)]. Below the radius associated with this loop, continuous rotational modes violate the kinematic bound derived from $\hbar/(mc)$, first noted in studies of minimal length scales [Mead(1964), Hossenfelder(2013)]. Consequently, angular change becomes pulse–driven rather than smooth, and the boundary supports discrete injections of angular defect rather than classical rotational eigenmodes.

By integrating radian–normalised angular defects, minimal closure structure, and pulse–based boundary dynamics into the Regge framework, the paper establishes a discrete geometric setting in which curvature, holonomy, and resonance spectra emerge directly from boundary geometry. This radian formulation offers a stricter, more metrologically coherent boundary description while retaining full compatibility with the classical Regge discretisation of curvature.

Keywords: Regge calculus; discrete geometry; radian normalisation; angular defect; holonomy; minimal geodesics; compact boundaries

Contents

1 Introduction

Regge calculus provides a discrete formulation of general relativity in which curvature is encoded through deficit angles concentrated on the simplicial hinges of a triangulated manifold [Regge(1961)]. This framework offers a coordinate–free representation of curvature and is widely employed in numerical relativity, lattice gravity, and geometric discretisations of gravitational actions. Despite its broad applicability, conventional Regge calculus inherits several geometric conventions from the smooth theory, including the treatment of angular quantities per cycle. These conventions introduce implicit factors of 2π into dihedral angles, holonomy, and boundary contributions to the classical action, obscuring the geometric provenance of angular terms and complicating comparisons across metrological frameworks.

The present work develops a radian–normalised refinement of Regge calculus for boundaries possessing a minimal non–contractible geodesic. On any compact surface with nontrivial fundamental group, the shortest representative of a non–contractible homotopy class exists and is itself a closed geodesic [Besse(1978)]. This loop forms the smallest physically meaningful angular cycle supported by the boundary. When its associated radius satisfies the inequality $r < \hbar/(mc)$, originally noted in early analyses of minimal length scales [Mead(1964)] and later examined in contemporary reviews of quantum–gravity phenomenology [Hossenfelder(2013)], continuous rotational motion becomes kinematically inadmissible. In such regimes, angular change cannot be implemented through smooth rotational eigenstates and must instead occur through finite, localised injections of angular defect.

This transition from smooth rotation to discrete angular pulses motivates a boundary refinement of Regge calculus in which angular increments are expressed per radian and curvature is recorded through radian–normalised defect elements rather than through 2π–scaled dihedral angles. The geometric structure arising from a minimal non–contractible loop naturally enforces this normalisation: the minimal loop provides a fixed geometric reference scale for angular change, and the kinematic bound at this scale promotes a pulse–driven mechanism for modifying boundary geometry. These pulses propagate along the compact boundary, decompose into discrete spectral modes, and interact with the simplicial structure by contributing quantised increments of angular defect to the Regge curvature.

By incorporating radian–normalised angular defects, minimal closure geometry, and pulse–driven boundary dynamics, the refined formulation developed here remains fully consistent with the classical Regge discretisation yet imposes a more transparent angular normalisation on the boundary degrees of freedom. This approach provides a geometric mechanism for the emergence of discrete curvature increments and clarifies the relation between angular holonomy and curvature accumulation, particularly on compact boundaries where the smallest geodesic loop imposes a strict geometric and kinematic constraint. The purpose of this paper is to formalise this boundary refinement, to analyse its geometric consequences, and to situate it within the broader class of discrete approaches to curvature.

2 Regge Foundations

Regge calculus provides a discrete geometric analogue of general relativity in which curvature is represented by angular defects located at the codimension–two hinges of a simplicial complex [Regge(1961)]. In this formulation, a smooth manifold is replaced by a piecewise–flat triangulation whose geometry is determined entirely by the edge lengths of its simplices. The metric is therefore implicit in the choice of simplicial structure, and curvature arises only where adjacent simplices meet in a manner that prevents the surrounding dihedral angles from summing to their flat–space values.

For a two–dimensional triangulated surface, curvature is concentrated at the vertices. If v is a vertex in the triangulation and $\theta_{T,v}$ denotes the interior angle at v within the triangle T, then the corresponding Regge deficit angle is defined by

$$\varepsilon_v = 2\pi - \sum_{T \ni v} \theta_{T,v}. \tag{1}$$

This quantity measures the deviation of the local geometry from flatness and provides a discrete representation of the Gaussian curvature associated with a dual cell surrounding v. When the triangulation is sufficiently refined, the sum of deficit angles converges to the integrated curvature predicted by the Gauss–Bonnet theorem, whose intrinsic formulation was established in classical differential geometry [Chern(1944)].

In higher dimensions, curvature is similarly represented by deficit angles at hinges of codimension two. In the physically relevant case of a $3 + 1$–dimensional spacetime discretised into four–simplices, curvature is concentrated on triangular faces. The Regge action is then given by

$$S_{\text{Regge}} = \frac{1}{8\pi G} \sum_h A_h \, \varepsilon_h, \tag{2}$$

where A_h denotes the area of hinge h and ε_h its associated deficit angle. Variation of this action with respect to the edge lengths yields a set of algebraic equations that approximate the Einstein field equations without introducing coordinate systems or connection coefficients.

The key geometric feature of Regge calculus is that curvature is not distributed continuously but is instead concentrated in finite angular deficits. This contrasts with the smooth formulation, where curvature is encoded in the Riemann tensor. The discrete approach captures the integral content of curvature while replacing differential operators with geometric sums over simplicial elements. As such, Regge calculus offers a natural framework for studying curvature in contexts where global geometric constraints, topological features, or minimal closed loops play a fundamental role.

This foundational structure makes Regge calculus particularly suitable for boundary refinement. Because the geometry is encoded in edge lengths and angular defects, any modification to the boundary normalisation or angular bookkeeping can be implemented without altering the bulk simplicial geometry. The radian–normalised boundary refinement developed in this work therefore sits naturally within the existing Regge framework, enhancing the interpretation of angular defect while preserving the classical structure of the simplicial discretisation.

3 Radian Boundary Refinement

Classical Regge calculus inherits the angular conventions of smooth differential geometry, in which angular quantities are implicitly expressed per cycle. Dihedral angles, boundary rotations, and holonomy contributions are therefore normalised with respect to 2π, reflecting the convention that a full rotation corresponds to a complete cycle. Although this convention is natural in the continuum, it obscures the geometric structure of deficit angles on compact boundaries where the smallest non–contractible loop supplies an intrinsic angular reference scale independent of the full 2π periodicity.

A radian–based refinement of the boundary geometry aims to remove this implicit normalisation by expressing all angular increments per radian rather than per cycle. This approach emphasises the differential character of angular change, aligns more closely with the local nature of curvature, and avoids the introduction of global factors that do not arise from the intrinsic geometry of the simplicial boundary. The classical Gibbons–Hawking–York term [York(1972), Gibbons and Hawking(1977)], for example, carries an explicit $1/(8\pi G)$ prefactor whose geometric interpretation is complicated by the implicit appearance of 2π in the associated angular integrations. A radian–normalised framework avoids such ambiguities by tying angular contributions directly to local geometric increments rather than to cycle–based normalisations.

The need for this refinement becomes especially clear on compact boundaries that admit a minimal non–contractible geodesic. As shown in the existence theory for closed geodesics on compact manifolds [Besse(1978)], such a loop provides the shortest angular cycle permitted by the boundary geometry. Its circumference, denoted c_0, supplies a natural angular scale that cannot be reduced without violating the topological constraints of the surface. When the radius associated with this loop satisfies the kinematic bound $r < \hbar/(mc)$, first identified in analyses of potential minimal length scales [Mead(1964)] and reinforced in later work on ultraviolet limitations [Hossenfelder(2013)], continuous angular motion becomes inaccessible. Angular change must therefore be implemented through finite increments that inject localised angular defect.

In a radian–normalised boundary geometry, these finite increments are expressed as discrete radian pulses rather than fractions of a 2π cycle. Each pulse contributes a radian–sized angular defect to the boundary simplices, altering the local dihedral configuration and modifying the corresponding Regge deficit angles. This pulse–driven structure aligns with the intrinsic nature of curvature in Regge calculus, where curvature is concentrated at simplicial hinges and recorded through finite angular deficits rather than through differential curvature tensors.

By adopting radian normalisation, the boundary refinement eliminates the dependence on cycle–based angular conventions and provides a more faithful representation of local angular geometry. This refinement clarifies the relation between boundary holonomy, angular defect, and curvature accumulation, preparing the framework for the introduction of discrete pulse dynamics and minimal–closure constraints developed in the subsequent sections. The approach remains fully compatible with the bulk simplicial geometry of Regge calculus but introduces a sharper and more metrologically coherent angular structure on compact boundaries.

4 Minimal Closure Derivations

Compact boundaries that admit a nontrivial fundamental group necessarily contain at least one non–contractible loop. Classical theorems in Riemannian geometry ensure that, within each homotopy class of closed curves on a compact surface, there exists a shortest representative that is itself a closed geodesic [Besse(1978)]. Let Σ denote a compact boundary surface with $\pi_1(\Sigma) \neq 0$, and let $[C]$ represent a non–contractible homotopy class. The minimal representative is given by

$$C_0 := \arg\min_{C' \sim C} \mathrm{Length}(C'), \tag{3}$$

and the length of this closed geodesic is defined as

$$\mathrm{Length}(C_0) = c_0. \tag{4}$$

This quantity serves as the minimal closure circumference of the boundary and provides a fixed geometric scale associated with the boundary's angular structure.

The corresponding radius of the minimal loop is $r_0 = c_0/(2\pi)$ when expressed using the standard 2π periodicity. However, the radian–normalised framework adopted in this work replaces the 2π cycle with local angular increments, so the radius is more appropriately understood as the geometric radius derived from the embedding of the closed geodesic itself. The key physical constraint associated with this radius arises from the kinematic bound on rotational motion for systems carrying angular momentum of order $\hbar$. In this context, the mass parameter m is interpreted as the characteristic mass scale of the boundary excitation responsible for supporting the minimal angular momentum state, providing the appropriate kinematic threshold for the loop. In this context, the mass parameter m is interpreted as the characteristic mass scale of the boundary excitation responsible for supporting the minimal angular momentum state, providing the appropriate kinematic threshold for the loop.

For a mass distribution m confined to a loop of radius r, classical rigid–body considerations yield the relation

$$L = mrv, \tag{5}$$

where L is the angular momentum and v is the tangential velocity. If the smallest nonzero angular momentum state is of order $\hbar$, as suggested by the quantum mechanical correspondence between angular momentum and action, then substituting $L \sim \hbar$ gives

$$v \sim \frac{\hbar}{mr}. \tag{6}$$

The relativistic constraint $v \leq c$ then imposes the kinematic bound

$$r \geq \frac{\hbar}{mc}. \tag{7}$$

This inequality was first exploited in early analyses proposing the existence of a minimal length scale in nature [Mead(1964)], and it continues to play a central role in reviews of quantum–gravity phenomenology that explore the consequences of ultraviolet completeness [Hossenfelder(2013)]. Equation (7) indicates that rotational degrees of freedom are kinematically forbidden when the radius of a loop falls below $\hbar/(mc)$.

Consequently, if the minimal non–contractible loop C_0 satisfies

$$r_0 < \frac{\hbar}{mc}, \tag{8}$$

continuous rotational motion along C_0 becomes inaccessible. Angular change must therefore be implemented through discrete increments that alter the geometry of the loop by injecting finite units of angular defect. These increments are interpreted as pulses rather than rotational eigenstates and provide the mechanism by which angular change is transmitted along the boundary.

In the radian–refined Regge framework, each pulse contributes a finite radian–normalised angular defect $\Delta\theta$ to the simplicial structure of the boundary, altering the corresponding deficit angles and modifying the local curvature. The minimal closure circumference c_0 thus plays a dual role: it is both a geometric invariant of the compact boundary and the threshold at which rotational dynamics transitions from continuous to pulse–based behaviour. This transition governs the discrete angular structure introduced in subsequent sections and ensures that curvature accumulation and holonomy remain consistent with the radian–normalised refinement of Regge calculus.

5 Discrete Pulse Geometry

The kinematic bound associated with the minimal non–contractible loop ensures that continuous rotational motion cannot be supported when the loop radius lies below the threshold $r = \hbar/(mc)$ [Mead(1964), Hossenfelder(2013)]. In such regimes, angular change must occur through discrete, finite adjustments of the boundary geometry rather than through smooth rotational eigenstates. This requirement introduces a pulse–driven mechanism for angular variation that is naturally compatible with the radian–normalised refinement of Regge calculus.

A pulse is defined as a localised increment of angular defect that propagates along the simplicial boundary. In contrast to continuous rotation, which distributes angular change smoothly around the loop, a pulse modifies the dihedral configuration at a specific hinge and subsequently shifts the distribution of deficit angles as it travels across adjacent simplices. Let $\Delta\theta$ denote the radian–normalised angular increment associated with a single pulse. When a pulse traverses a hinge h with interior angle $\theta_{T,h}$ in a triangle T, the pulse introduces a modified interior angle $\theta'_{T,h} = \theta_{T,h} + \Delta\theta$ and thereby alters the corresponding Regge deficit angle.

For a vertex v on a triangulated boundary, the updated deficit angle under the passage of a pulse is therefore given by

$$\varepsilon'_v = 2\pi - \sum_{T \ni v} \theta'_{T,v} = \varepsilon_v - \sum_{T \ni v} \Delta\theta_{T,v}, \tag{9}$$

where each increment $\Delta\theta_{T,v}$ records the contribution of the pulse to the interior angle at v within triangle T. Because $\Delta\theta$ is expressed per radian, the resulting modification to the deficit angle is independent of any cycle–based normalisation and is determined solely by the local effect of the pulse.

The propagation of a pulse along a compact loop imposes additional structure. As the pulse travels, it can be decomposed into eigenmodes of the angular Laplacian on the loop. For a loop of radius R, parametrised by an angular coordinate θ, a scalar field $\phi(\theta, t)$ describing the pulse satisfies the one–dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{v_{\mathrm{p}}^2}{R^2} \frac{\partial^2 \phi}{\partial \theta^2}, \tag{10}$$

where v_{p} is the characteristic propagation speed. Periodicity enforces the spectral decomposition

$$\phi(\theta, t) = \sum_{n \in \mathbb{Z}} A_n \exp(in\theta - i\omega_n t), \tag{11}$$

with discrete frequencies

$$\omega_n = \frac{v_{\mathrm{p}}}{R} |n|. \tag{12}$$

These eigenmodes form the natural resonance spectrum of angular pulses on the minimal loop. The compactness of the boundary ensures that only integer–labelled modes are allowed, and the minimal closure geometry restricts the possible values of R.

Each pulse thus corresponds to a finite superposition of such modes, and its interaction with the simplicial boundary modifies the local angular structure in quantised increments. The radian–normalised formulation ensures that these increments carry a clear geometric interpretation, directly altering the Regge deficit angles without reliance on hidden 2π conventions. This pulse–driven geometry provides the operational mechanism through which angular change is implemented on compact boundaries where continuous rotation is kinematically prohibited.

The resulting discrete pulses supply the boundary degrees of freedom required for the accumulation of curvature and the generation of holonomy. Because curvature in Regge calculus is concentrated in finite deficits rather than distributed continuously, the pulse–based mechanism aligns precisely with the intrinsic nature of Regge curvature, offering a geometrically coherent method for encoding discrete angular modifications at the smallest boundary scale. This pulse description is introduced as a geometric representation of discrete angular increments within the radian–normalised boundary framework, rather than as a physical dynamical process; no claim is made regarding the underlying microscopic mechanism beyond its role in the discrete angular bookkeeping of the boundary geometry.

Worked Example: Three–Triangle Boundary Fan

To illustrate the effect of a radian–normalised pulse on the simplicial boundary, consider a minimal configuration in which three boundary triangles meet at a vertex v. Let the interior angles at v be denoted

$$\theta_1, \quad \theta_2, \quad \theta_3,$$

so that the initial Regge deficit at v is

$$\varepsilon_v = 2\pi - (\theta_1 + \theta_2 + \theta_3).$$

Suppose a single radian–normalised pulse contributes a finite angular increment $\Delta\theta$ to the interior angle of triangle T_1, modifying it to

$$\theta'_1 = \theta_1 + \Delta\theta.$$

The updated deficit angle becomes

$$\varepsilon_v' = 2\pi - (\theta_1' + \theta_2 + \theta_3) = \varepsilon_v - \Delta\theta.$$

Thus, a radian–normalised pulse changes the local curvature at v by a finite amount proportional to its angular increment. No reference to a 2π cycle is required: the modification arises solely from the local contribution to a single interior angle. This simple configuration demonstrates explicitly how pulse–based increments enter the Regge bookkeeping and how discrete angular changes modify curvature at the boundary.

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6 Holonomy and Curvature Quantisation

In the continuous theory of differential geometry, the holonomy accrued by parallel transport around a closed loop is determined by the integrated curvature enclosed by that loop. In the discrete setting of Regge calculus, this relation is implemented through the accumulation of deficit angles [Regge(1961)]. When a loop Γ encloses a set of vertices $v \in D(\Gamma)$ on a triangulated boundary, the holonomy angle Ω_Γ is approximated by

$$\Omega_\Gamma \approx \sum_{v \in D(\Gamma)} \varepsilon_v, \tag{13}$$

where each ε_v denotes the Regge deficit at vertex v. This expression mirrors the integral form of the Gauss–Bonnet relation, whose intrinsic formulation was established in classical differential geometry [Chern(1944)].

The radian–normalised refinement developed in this work modifies this structure only at the boundary level. Because curvature in Regge calculus is encoded discretely, any radian–based adjustment to the boundary introduces finite angular increments that directly alter the deficit angles. When a pulse of radian–normalised angular defect propagates across the boundary, the corresponding modification to each interior angle appears as a finite increment $\Delta\theta$, and the resulting change in holonomy around a closed loop becomes

$$\Delta\Omega_\Gamma = - \sum_{v \in D(\Gamma)} \sum_{T \ni v} \Delta\theta_{T,v}. \tag{14}$$

Because the pulse carries an angle normalised to a single radian rather than to a fraction of a 2π cycle, the induced holonomy increment is expressed without reference to global angular scaling. This refinement ensures that holonomy is tied directly to the local mechanics of angular defect rather than to an implicit periodicity inherited from the smooth theory.

When the closed loop Γ is chosen to be the minimal non–contractible geodesic C_0, its fixed circumference imposes additional structure on the holonomy. The compactness of the loop forces all pulses travelling along it to assume discrete eigenmodes of the form

$$\exp(in\theta - i\omega_n t)\,, \qquad n \in \mathbb{Z}, \tag{15}$$

where the integer n labels the angular resonance mode. Each such mode contributes a quantised holonomy increment that depends on n and on the radian–normalised angular defect associated with the pulse. These contributions accumulate discretely, yielding a holonomy spectrum of the form

$$\Omega_\Gamma(n) = \Omega_0 + n\,\Delta\Omega, \tag{16}$$

where $\Delta\Omega$ is the holonomy increment induced by a single pulse of angular defect and Ω_0 is the base holonomy associated with the unperturbed simplicial configuration.

This spectrum represents a discrete counterpart to curvature quantisation at the boundary. Because each pulse introduces a finite radian–normalised defect, and because the boundary supports only integer–indexed angular modes, the resulting holonomy increments form a linear lattice parametrised by the integer n. The structure is a direct consequence of the radian refinement and the minimal closure geometry rather than an externally imposed quantisation rule.

The pulse–induced holonomy structure therefore provides a mechanism for discrete curvature accumulation. As pulses traverse the minimal loop, they inject radian–normalised angular defect into the boundary simplices, altering the deficit angles in quantised steps. The resulting holonomy increments correspond to discrete curvature elements whose values depend only on the geometry of the minimal loop and the pulse structure introduced earlier. This mechanism integrates seamlessly with the classical Regge framework and provides a geometrically coherent route to curvature quantisation on compact boundaries.

7 Discussion

The boundary refinement developed in this work provides a radian–normalised description of angular geometry that integrates directly with the simplicial foundations of Regge calculus. By replacing implicitly cycle–based angular conventions with local radian increments, the refinement removes global factors that do not arise from the intrinsic geometry of the boundary and reveals the discrete nature of angular modification when the minimal loop circumference approaches the kinematic limit. This structural shift clarifies the geometric interpretation of deficit angles, holonomy, and curvature accumulation in settings where the smallest non–contractible loop plays a fundamental dynamical role.

A central outcome of the refinement is the transition from smooth rotational degrees of freedom to discrete pulse–driven angular change. When the radius of the minimal loop satisfies the bound $r < \hbar/(mc)$, the boundary cannot support classical rotational eigenstates [Mead(1964), Hossenfelder(2013)]. Instead, angular modification must occur through finite injections of radian–normalised defect. These pulses propagate along the compact loop, decompose into discrete integer–labelled modes, and modify the local deficit angles in quantised increments. The resulting behaviour aligns naturally with the discrete curvature structure inherent to Regge calculus, where curvature is stored in angular deficits rather than in continuous tensors.

The refinement also clarifies the geometric origin of holonomy increments on compact boundaries. Because each pulse contributes a radian–normalised defect to one or more hinges, the holonomy accumulated around a closed loop becomes quantised in integer multiples of a fixed increment. This feature does not rely on external assumptions about quantisation but arises directly from the combined effects of compactness, minimal closure geometry, and the discrete nature of Regge curvature. The resulting lattice of holonomy values reflects the interplay between the boundary's spectral structure and the radian–based angular increments introduced by pulses.

In broader terms, the radian refinement strengthens the metrological consistency of Regge calculus on compact boundaries. By tying angular increments to local radian units rather than to cycle–based conventions, the refinement ensures that curvature contributions, boundary actions, and holonomy calculations rest on a geometrically transparent foundation. This contributes to a more coherent boundary description, particularly in regimes where the minimal loop sets a fundamental geometric scale and where discretised angular dynamics dominate.

The discrete pulse structure also suggests new avenues for analysing curvature on highly compact or topologically constrained boundaries. Because the minimal loop plays a dual role as a geometric invariant and as a dynamical threshold, it becomes a natural candidate for anchoring discrete boundary models in contexts where continuous rotational symmetry is restricted. The radian–normalised framework provides a mathematically controlled method for exploring such regimes without altering the bulk structure of Regge calculus, preserving the established geometric content while sharpening the boundary representation.

Overall, the discussion highlights how the radian boundary refinement complements the classical simplicial framework, offering both greater transparency in angular normalisation and a natural mechanism for discrete curvature accumulation. The refinement remains compatible with the underlying structure of Regge geometry and enhances its applicability to compact boundaries where minimal closure constraints play an essential role.

8 Conclusion

This paper has developed a radian–normalised refinement of Regge calculus for compact boundaries supporting a minimal non–contractible geodesic. The refinement addresses a structural feature of the classical framework: the reliance on cycle–based angular conventions inherited from smooth differential geometry. By expressing angular increments per radian rather than per cycle, the boundary geometry acquires a more transparent and locally grounded interpretation of dihedral angles, deficit angles, and curvature accumulation.

The minimal closure loop provides the geometric foundation for this refinement. Classical existence theorems guarantee that such a loop is present on any compact boundary with nontrivial fundamental group [Besse(1978)]. When its radius approaches the kinematic threshold $\hbar/(mc)$, continuous rotational modes become inaccessible [Mead(1964), Hossenfelder(2013)], and angular change must proceed through discrete pulses rather than smooth rotation. These pulses inject finite radian–normalised angular defect into the simplicial structure and propagate along the compact loop as superpositions of integer–labelled resonance modes.

The interaction between discrete pulses and Regge deficit angles yields a quantised holonomy structure. Each pulse induces a finite modification to the dihedral configuration at the boundary hinges, and the accumulated curvature around a closed loop becomes quantised in integer multiples of a fixed holonomy increment. This behaviour arises naturally from the discrete geometry of the boundary and from the radian normalisation adopted in the refinement, without the need for additional postulates.

The resulting framework preserves the bulk geometric content of Regge calculus while offering a sharper boundary representation that is metrologically consistent and geometrically precise. By clarifying the angular structure associated with compact boundaries and by providing a discrete mechanism for curvature accumulation, the radian–normalised refinement enhances the utility of Regge calculus in regimes where minimal closure constraints restrict the available angular dynamics. The approach therefore contributes to a more complete understanding of discrete curvature on compact boundaries and establishes a foundation for further geometric investigations based on radian–normalised angular structures.

A Figures

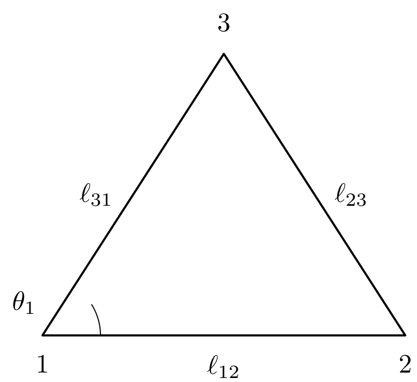


Figure 1. Local triangle geometry: edge lengths ℓ_{ij} determine interior angles such as θ_1 via the law of cosines.

$$\varepsilon_v = 2\pi - \sum_i \theta_i$$

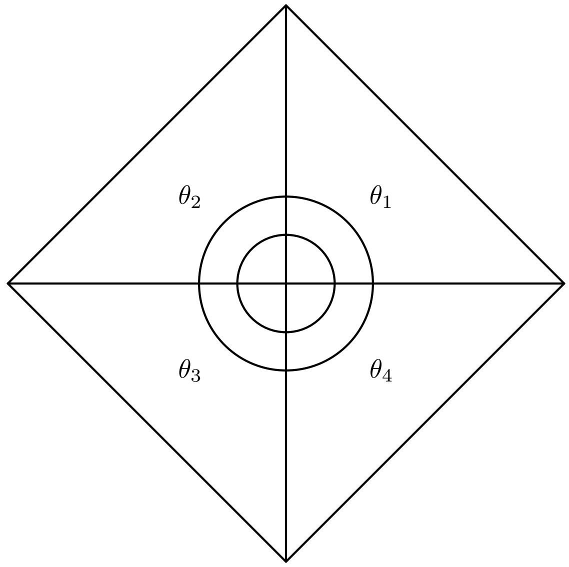


Figure 2. Curvature at a vertex v in Regge calculus: the deficit angle ε_v measures how far the sum of interior angles at v falls short of 2π.

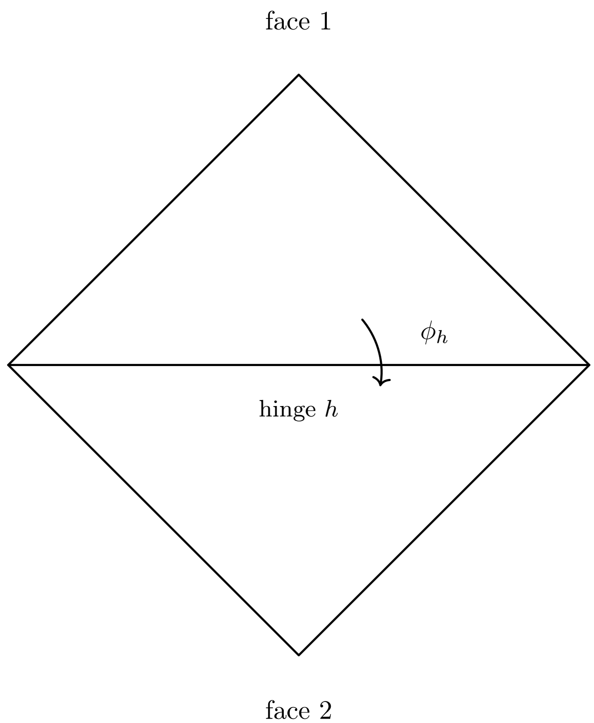
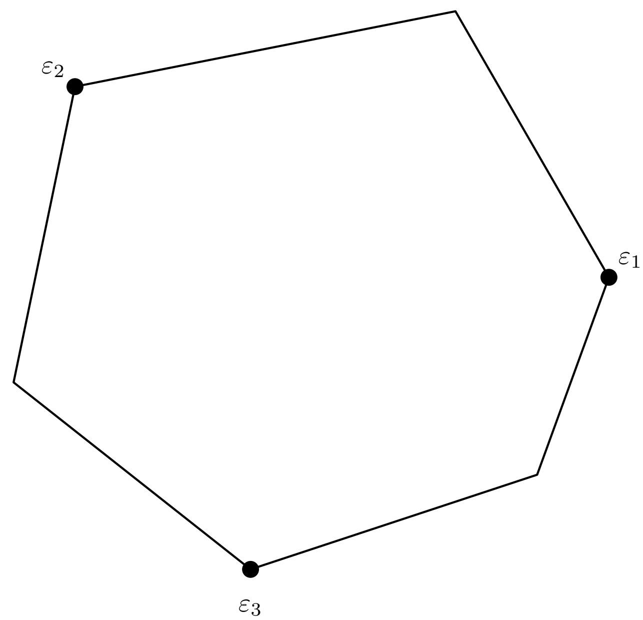


Figure 3. Dihedral angle ϕ_h between two faces meeting along an edge h.



$$\Omega_\Gamma \approx \sum_{v\in\Gamma} \varepsilon_v$$

Figure 4. Holonomy around a closed loop Γ: the accumulated angular deviation is approximated by the sum of deficit angles at vertices along Γ.

B Symbol Glossary

Symbol	Definition	Units
A_h	Area of hinge h in Regge calculus	m^2
A_n	Amplitude of angular resonance mode n	–
c	Speed of light in vacuum	$m\,s^{-1}$
c_0	Minimal closure circumference of the boundary loop	m
$\Delta\theta$	Radian–normalised angular increment (pulse contribution)	rad
G	Newtonian gravitational constant	$m^3\,kg^{-1}\,s^{-2}$
L	Angular momentum associated with the boundary loop	$kg\,m^2\,s^{-1}$
ℓ_k	Regge edge length between vertices	m
m	Mass associated with boundary rotational mode or pulse constraint	kg
m_{P}	Planck mass $\sqrt{\hbar c/G}$	kg
n	Integer label of angular resonance mode	–
Ω_Γ	Holonomy angle accumulated around closed loop Γ	rad
ω_n	Angular frequency of mode n	$rad\,s^{-1}$
r	Radius of boundary loop or geodesic	m
r_0	Radius associated with minimal closure loop C_0	m
$r_{\min}$	Kinematic threshold radius $\hbar/(mc)$ below which rotation is forbidden	m
S_{Regge}	Regge action expressing curvature through hinge deficits	J s
t	Time coordinate	s
v	Tangential speed along boundary loop	$m\,s^{-1}$
v_{p}	Pulse propagation speed along compact boundary	$m\,s^{-1}$
ε_h	Regge deficit angle at hinge h	rad
ε_v	Regge deficit angle at vertex v (for 2D triangulations)	rad
$\phi(\theta, t)$	Scalar field describing angular pulse profile on loop	–
θ	Angular coordinate on minimal loop or compact boundary	rad

Acknowledgments

This work draws upon established results in discrete geometry, classical analyses of curvature concentration, and foundational studies linking minimal length scales to kinematic constraints. The refinement presented here benefits from prior developments in Regge calculus, intrinsic formulations of the Gauss–Bonnet relation, and boundary contributions to gravitational action. The author acknowledges the broader mathematical and physical literature that has shaped the theoretical context in which this radian–normalised boundary framework has been constructed.

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